

Arithmetic functions commutable with sums of squares

Jungin Lee

Abstract. In this note, we characterize all functions $f : \mathbb{N} \rightarrow \mathbb{C}$ such that $f(x_1^2 + \cdots + x_k^2) = f(x_1)^2 + \cdots + f(x_k)^2$, where $k \geq 3$ and x_1, \dots, x_k are positive integers.

1 Introduction

In 1996, Chung [2] classified multiplicative functions satisfying $f(m^2 + n^2) = f(m)^2 + f(n)^2$ for all $m, n \in \mathbb{N}$. Bašić [1] generalized this result to arbitrary arithmetic functions. Recently, Park ([3], [4]) proved that for every integer $k \geq 3$, a multiplicative function f satisfies $f(x_1^2 + \cdots + x_k^2) = f(x_1)^2 + \cdots + f(x_k)^2$ for all $x_1, \dots, x_k \in \mathbb{N}$ is an identity function. We generalize Park's result to arithmetic functions, as Bašić generalized Chung's result.

2 Results

Theorem. Let $k \geq 3$ be an integer. If a function $f : \mathbb{N} \rightarrow \mathbb{C}$ satisfies $f(x_1^2 + \cdots + x_k^2) = f(x_1)^2 + \cdots + f(x_k)^2$ for every $x_1, \dots, x_k \in \mathbb{N}$, then one of the following holds:

- (1) $f \equiv 0$
 - (2) $f(n) = \begin{cases} n & (\exists x_1, \dots, x_k \in \mathbb{N} \ n = x_1^2 + \cdots + x_k^2) \\ \pm n & (\text{otherwise}) \end{cases}$
 - (3) $f(n) = \begin{cases} \frac{1}{k} & (\exists x_1, \dots, x_k \in \mathbb{N} \ n = x_1^2 + \cdots + x_k^2) \\ \pm \frac{1}{k} & (\text{otherwise}) \end{cases}$
- (In (2) and (3), each \pm sign is independent.)

Proof. Denote $f(n)^2$ by A_n . If $k = 3$, then

$$A_{n+2} + A_{n-2} + A_1 = f(2n^2 + 9) = 2A_n + A_3 \quad (n \geq 3) \quad (1)$$

so it is enough to show that (1), (2) or (3) holds for $n \leq 4$. If $k \geq 4$, then

$$A_{n+1} + A_{n-1} + 2A_2 + (k-4)A_1 = f(2n^2 + k + 6) = 2A_n + (k-3)A_1 + A_3 \quad (n \geq 3) \quad (2)$$

so it is enough to show that (1), (2) or (3) holds for $n \leq 3$. For convenience, we denote A_1, \dots, A_5 by A, B, C, D and E .

Case I. $k = 3$

$$C = (3f(1)^2)^2 = 9A^2 \quad (3a)$$

$$C + 2D = f(3)^2 + 2f(4)^2 = f(41) = f(6)^2 + f(1)^2 + f(2)^2 = (2A + B)^2 + A + B \quad (3b)$$

$$2A + (3B)^2 = 2f(1)^2 + f(12)^2 = f(146) = f(11)^2 + f(3)^2 + f(4)^2 = (2A + C)^2 + C + D \quad (3c)$$

$$\begin{aligned} (A + 2C)^2 + C + A &= f(19)^2 + f(3)^2 + f(1)^2 = f(371) \\ &= f(17)^2 + f(9)^2 + f(1)^2 = (2B + C)^2 + (A + 2B)^2 + A \end{aligned} \quad (3d)$$

$$E + 2A = f(5)^2 + 2f(1)^2 = f(27) = 3f(3)^2 = 3C \quad (3e)$$

$$E + 2B = f(5)^2 + 2f(2)^2 = f(33) = f(1)^2 + 2f(4)^2 = A + 2D \quad (3f)$$

$$\begin{aligned} (2A + C)^2 + A + B &= f(11)^2 + f(1)^2 + f(2)^2 = f(126) \\ &= f(9)^2 + f(6)^2 + f(3)^2 = (A + 2B)^2 + (2A + B)^2 + C \end{aligned} \quad (3g)$$

From the equations (3a), (3b), (3e) and (3f), we can obtain $(B - 4A)(B + 8A - 1) = 0$.

(i) $B = 4A$: If we substitute $B = 4A$ and $C = 9A^2$ to the equation (3d), we obtain $27A^2(A - 1)(9A + 5) = 0$. If $A = 0$, $B = C = 0$ and $D = 0$ by (3b). If $A = 1$, $B = 4$, $C = 9$ and $D = 16$ by (3b). If $A = -\frac{5}{9}$, $B = -\frac{20}{9}$, $C = \frac{25}{9}$ and this contradicts to (3g).

(ii) $B = 1 - 8A$: If we substitute $B = 1 - 8A$ and $C = 9A^2$ to the equations (3d) and (3g), we obtain $(9A - 1)(27A^3 + 39A^2 - 52A + 8) = 0$ and $(9A - 1)(9A^3 + 5A^2 - 29A + 4) = 0$, respectively. Thus $A = \frac{1}{9}$, $B = C = \frac{1}{9}$, and $D = \frac{1}{9}$ by (3b).

Case II. $k = 4$

$$\begin{aligned} (A + 3B)^2 + (2A + 2B)^2 + A + B &= f(13)^2 + f(10)^2 + f(1)^2 + f(2)^2 \\ &= f(274) = f(16)^2 + f(4)^2 + 2f(1)^2 = (4B)^2 + (4A)^2 + 2A \end{aligned} \quad (4a)$$

$$\begin{aligned} (4B)^2 + 3B &= f(16)^2 + 3f(2)^2 = f(268) \\ &= f(13)^2 + 2f(7)^2 + f(1)^2 = (A + 3B)^2 + 2(3A + B)^2 + A \end{aligned} \quad (4b)$$

$$(3A + B)^2 + 3A = f(7)^2 + 3f(1)^2 = f(52) = 3f(4)^2 + f(2)^2 = 48A^2 + B \quad (4c)$$

$$E + 3A = f(5)^2 + 3f(1)^2 = f(28) = 3f(3)^2 + f(1)^2 = 3C + A \quad (4d)$$

$$E + 3B = f(5)^2 + 3f(2)^2 = f(37) = 2f(4)^2 + f(1)^2 + f(2)^2 = 32A^2 + A + B \quad (4e)$$

From the equation (4a), we can obtain $(A - B)(11A - 3B + 1) = 0$.

(i) $B = A$: If we substitute $B = A$ to the equation (4b), we obtain $32A^2 = 2A$, so A is 0 or $\frac{1}{16}$. By the equations (4d) and (4e), $(A, B, C) = (0, 0, 0), (\frac{1}{16}, \frac{1}{16}, \frac{1}{16})$.

(ii) $B = \frac{11A+1}{3}$: If we substitute $B = \frac{11A+1}{3}$ to the equations (4b) and (4c), we obtain $(A - 1)(160A + 14) = 0$ and $(A - 1)(16A - 1) = 0$, respectively. Thus $A = 1$, $B = 4$ and $C = 9$ by (4d) and (4e).

Case III. $k \geq 5$

$$2(A - B)^2 + A + 2B = 3C \quad (5a)$$

$$5A + 3C = 8B \quad (5b)$$

From $(kA)^2 + ((k-2)A + 2B)^2 + 2B + (k-4)A = f(k)^2 + f(k+6)^2 + 2f(2)^2 + (k-4)f(1)^2 = f(2k^2 + 13k + 40) = 2f(k+3)^2 + 3f(3)^2 + (k-5)f(1)^2 = 2((k-1)A + B)^2 + 3C + (k-5)A$, we obtain (5a). From $(D + 4A) + (A + 3C) = f(20) + f(28) = 5B + (D + 3B)$, we obtain (5b). From the equations (5a) and (5b), we obtain $2(A - B)(A - B + 3) = 0$.

(i) $B = A$: By the equation (5b), $C = A$ and by the equation (2), $A_n = A$ for every $n \in \mathbb{N}$. Then $A = A_k = (kf(1)^2)^2 = k^2A^2$, so A is 0 or $\frac{1}{k^2}$.

(ii) $B = A + 3$: By the equation (5b), $C = A + 8$ and by the equation (2), $A_n - n^2 = A - 1$ for every $n \in \mathbb{N}$. Then $((k-1)A + (A+3))^2 - (k+3)^2 = A_{k+3} - (k+3)^2 = A_k - k^2 = (kA)^2 - k^2$, so $A = 1$. \square

References

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Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Korea
e-mail: moleculesum@postech.ac.kr